CS 599: Rounding Techniques in Approximation Algorithms

Homework 1: Intro and Independent Randomized Rounding

Fall 2024

1 Problem 1: Vertex Cover

1. Write the natural integer programming relaxation for Vertex Cover and the corresponding relaxed linear program.
2. Prove that if x is an extreme point solution to this LP, then $x_v \in \{0, \frac{1}{2}, 1\}$ for all $v \in V$.
3. Give a $\frac{3}{2}$ approximation for Vertex Cover on planar graphs. Show a matching lower bound on the integrality gap.
4. Given an example that shows the greedy algorithm for vertex cover (iteratively picking the max degree vertex and deleting it and its edges from the graph) has approximation ratio $\Omega(\log n)$.
5. *(Bonus question)* Give a lower bound showing that the Vertex Cover LP with triangle constraints (see Lecture 1) still has an integrality gap of 2 (up to lower order terms).

2 Problem 2: Chernoff Bounds

1. Prove that the congestion for the multi-commodity flow problem we discussed in Lecture 2 can be improved to $O(\log n / \log\log n)$.
2. Prove that the maximum degree of a vertex in a uniformly random spanning tree on the complete graph is $O(\log n / \log\log n)$ with high probability. You may use the Prüfer code.

3 Problem 3: Set Cover

Set cover is a generalization of vertex cover. Here we are given elements $E = \{e_1, \ldots, e_n\}$ and some sets $S_1, \ldots, S_m \subseteq E$ with non-negative weights $w_1, \ldots, w_m \in \mathbb{R}_{\geq 0}$. Our goal is to select a collection of sets $\mathcal{S}$ with minimum cost that covers all the elements, i.e. we should have $\bigcup_{S \in \mathcal{S}} S_i = E$.

1. Write the natural integer programming relaxation for set cover and the corresponding relaxed linear program.
2. Prove that the probability a given element is covered by including each set S_i in $\mathcal{S}$ independently with probability x_i is at least $1 - 1/e$.
3. Give a $O(\log n)$ approximation for set cover that works with probability at least $1 - 1/n$.
4. Prove that the integrality gap of the LP for set cover is at least $(1 - \epsilon) \ln n$ for every $\epsilon > 0$. **Hint:** Use a random instance.
5. *(Bonus question)* Give a deterministic construction with gap $\Omega(\log n)$.